Nambu quantum mechanics on discrete 3-tori

2009 J. Phys. A: Math. Theor. 42 275201

(http://iopscience.iop.org/1751-8121/42/27/275201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 275201 (12pp) doi:10.1088/1751-8113/42/27/275201

Nambu quantum mechanics on discrete 3-tori

M Axenides[1], **E G Floratos**[2] **and S Nicolis**[3]

[1] National Research Center 'Demokritos', 15310 Aghia Paraskevi, Athens, Greece
[2] Nuclear and Particle Physics Section, University of Athens, 15771 Athens, Greece
[3] CNRS—Laboratoire de Mathématiques et Physique Théorique (UMR 6083) Fédération Dénis Poisson (FR 9164) Université de Tours 'François Rabelais', Parc Grandmont, 37200 Tours, France

E-mail: axenides@inp.demokritos.gr, mflorato@physics.uoa.gr and Stam.Nicolis@lmpt.univ-tours.fr

Received 17 January 2009, in final form 5 May 2009
Published 11 June 2009
Online at stacks.iop.org/JPhysA/42/275201

Abstract
We propose a quantization of linear, volume preserving, maps on the discrete and finite 3-torus $\mathbb{T}_N^3$ represented by elements of the group $SL(3, \mathbb{Z}_N)$. These flows can be considered as special motions of the Nambu dynamics (linear Nambu flows) in the three-dimensional toroidal phase space and are characterized by invariant vectors $\boldsymbol{a}$ of $\mathbb{T}_N^3$. We quantize all such flows, which are necessarily restricted on a planar two-dimensional phase space, embedded in the 3-torus, transverse to the vector $\boldsymbol{a}$. The corresponding maps belong to the little group of $\boldsymbol{a} \in SL(3, \mathbb{Z}_N)$, which is an $SL(2, \mathbb{Z}_N)$ subgroup. The associated linear Nambu maps are generated by a pair of linear and quadratic Hamiltonians (Clebsch–Monge potentials of the flow) and the corresponding quantum maps realize the metaplectic representation of $SL(3, \mathbb{Z}_N)$ on the discrete group of three-dimensional magnetic translations, i.e. the non-commutative 3-torus with a deformation parameter the Nth root of unity. Other potential applications of our construction are related to the quantization of deterministic chaos in turbulent maps as well as to quantum tomography of three-dimensional objects.

PACS numbers: 03.65.Fd, 02.40.−k

1. Introduction

Recently, due to the progress in understanding the dynamics of the low-energy effective field theories for systems of multiple membranes, in analogy with the AdS/CFT correspondence [1], new algebraic structures, the metric 3-algebras, which are related to the quantization problem of Nambu 3-brackets [2–7], have attracted considerable interest [8–13].

In [13], we discussed in detail the relation between metric 3-algebras and Nambu 3-brackets, and we proposed a specific quantization method inspired by the work of Takhtajan [3].

In his classic paper [2], Nambu introduced a generalization of classical mechanics, where the role of canonical transformations of Hamiltonian mechanics is played by the general volume-preserving diffeomorphism group of a manifold of any dimension, considered as the corresponding phase space. For example, in three-dimensional Euclidean space, we consider incompressible flows and the particle trajectory flow equations are the Nambu dynamical equations: first order in time differential equations that generalize Hamilton's equations of motion.

In the following section, we recall that, in this case, there are two Hamiltonians H_1, H_2, corresponding to the Clebsch–Monge potentials of the flow, which are conserved and their constant values define a double family of intersecting surfaces, the intersections of which define the trajectories of the (test) particles carried by the flow. The interpretation we adopted in [13] is that the surfaces defined by the second potential are conventional, two-dimensional, phase spaces, foliating the 3-space, and the first potential defines conventional Hamiltonian mechanics on these phase spaces. There is a common, induced, Poisson bracket on the two-dimensional phase spaces embedded in $\mathbb{R}^3$, and the Nambu equations on these phase spaces take the standard Hamiltonian form.

The quantum mechanics of this system must follow the quantization of the induced Poisson structure, which depends on the symplectic structure on the H_2 surfaces. If the H_2 Hamiltonian is linear and H_1 is quadratic, we have an incompressible linear flow in $\mathbb{R}^3$.

In the present work, we consider the classical discretization of these linear flows(maps) in toroidal discrete three-dimensional phase space and their quantization. The corresponding quantum three-dimensional phase space is a non-commutative 3-torus with rational values of the non-commutativity parameter [21]. The classical linear maps of $SL(3, \mathbb{Z}_N)$ are related to strong arithmetic (deterministic) chaos [22] and can be considered as discrete toy models for turbulence on $\mathbb{T}^3$ [23]. These considerations and the possible physical interpretation of their quantized version will be discussed elsewhere.

The plan of the paper is as follows.

In section 2 we recall the formulation of Nambu dynamics in $\mathbb{R}^3$ and $\mathbb{T}^3$; in section 3 we consider the deterministic, chaotic, linear maps, analogs of the Arnold cat maps (but in three dimensions), which are elements of $SL(3, \mathbb{R})$ and introduce their Lie algebra. In section 4, we pass to the discretized 3-torus and consider corresponding maps, which are the elements of $SL(3, \mathbb{Z}_N)$ and, indeed, belong to the little group, $SL(2, \mathbb{Z}_N)$, of the invariant vectors normal to the planes of the flow. In section 5, we consider the non-commutative, rational, 3-torus and we construct the Heisenberg–Weyl group that corresponds to the linear Nambu flows as well as the associative, quantum, 3-algebra for the foliation of the 3-torus by the normal vectors.

In section 6, we present the quantization of these maps. The quantum maps (realized by unitary $N \times N$ matrices) are constructed explicitly by imposing the (metaplectic) representation of $SL(3, \mathbb{Z}_N)$, induced by the little group $SL(2, \mathbb{Z}_N)$.

We end with our conclusions, interpretation of our results and discuss some emergent applications.

2. Nambu mechanics in $\mathbb{R}^3$ and $\mathbb{T}^3$

In his classic paper [2], Y Nambu generalized classical Hamilton–Poisson mechanics by considering arbitrary dimensions for the phase space, replacing the canonical transformation symmetry by the volume-preserving diffeomorphisms [8]. In the particular case of three-dimensional, flat, phase space, one needs *two* 'Hamiltonian' functions, and the Nambu equations of motion take the form

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \{x^i, H_1, H_2\}, \tag{1}$$

$i = 1, 2, 3$, with initial conditions $x^i(t = 0) \equiv x^i(0)$.

The 3-bracket for functions, $f, g, h \in \mathscr{C}^\infty(\mathbb{R}^3)$ is defined as

$$\{f, g, h\} \equiv \varepsilon^{ijk}\partial_i f \partial_j g \partial_k h. \tag{2}$$

The Nambu bracket is invariant under general coordinate transformations, which preserve the volume, i.e. for $y^i = u^i(\boldsymbol{x})$ with $J = \det \partial u^i/\partial x^j = 1$. These transformations form the volume-preserving diffeomorphism group, $\mathrm{SDiff}(\mathbb{R}^3)$, under the composition of mappings [14]. For infinitesimal transformations

$$x^i \to x^i + v^i(\boldsymbol{x}), \tag{3}$$

with $v^i(\boldsymbol{x})$ being a divergenceless vector field. This transformation defines the flow

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{x}), \tag{4}$$

with corresponding generators, $X(\boldsymbol{v}) \equiv -v^i\partial^i$. They form the Lie algebra,

$$[X(\boldsymbol{v}), X(\boldsymbol{u})] = X(\boldsymbol{w}), \tag{5}$$

where

$$w^i \equiv \varepsilon^{ijk}\partial^j((\boldsymbol{v} \times \boldsymbol{u})^k).$$

The right-hand side of equation (4) can be written in terms of two Clebsch–Monge potentials [15], H_1 and H_2:

$$v^i = \varepsilon^{ijk}\partial^j H_1 \partial^k H_2. \tag{6}$$

The Lie algebra of $\mathrm{SDiff}(\mathbb{R}^3)$, in the Clebsch–Monge gauge, becomes

$$[X(H_1, H_2), X(H_3, H_4)] = X(X(H_1, H_2)H_3, H_4) + X(H_3, X(H_1, H_2)H_4). \tag{7}$$

Note that $X(H_1, H_2)H_3 = -\{H_1, H_2, H_3\}$, i.e. they realize the Nambu bracket.

From the above we see that the Nambu equations of motion describe incompressible flows in $\mathbb{R}^3$ and the solutions represent the integral curves of the flow $v^i(H_1, H_2)$.

The 3-bracket has certain interesting properties [3]: It

- is *multilinear* in f, g, h,
- is *antisymmetric* in f, g, h,
- has the *Leibniz* property:

$$\{f_1 f_2, g, h\} = f_1\{f_2, g, h\} + \{f_1, g, h\}f_2, \tag{8}$$

- satisfies the *fundamental identity,*

$$\{\{f_1, f_2, f_3\}, f_4, f_5\} + \{f_1, \{f_4, f_2, f_3\}, f_5\} + \{f_1, f_4, \{f_5, f_2, f_3\}\} \\ = \{\{f_1, f_4, f_5\}, f_2, f_3\}. \tag{9}$$

The fundamental identity can be proved by applying both sides of equation (7) with $f_i = H_i, i = 1, \ldots, 4$, on a function f_5, where $f_i \in \mathscr{C}^\infty(\mathbb{R}^3), i = 1, \ldots, 5$.

We can obtain Liouville's equation, for an arbitrary observable, that does not depend explicitly on time:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H_1, H_2\}. \tag{10}$$

This equation implies the conservation of the 'Hamiltonians' H_1 and H_2 under the flow and, thus, the particle's trajectory lies on the intersection of the two surfaces in $\mathbb{R}^3$, defined by H_1 and H_2, given the initial conditions, $\boldsymbol{x}(0)$. Its formal solution may be written as

$$f(\boldsymbol{x}) = \mathrm{e}^{-tX(H_1,H_2)} f(\boldsymbol{x}(0)). \tag{11}$$

We shall need later the toroidal compactification of $\mathbb{R}^3$, $\mathbb{T}^3 \equiv \mathbb{R}^3/\mathbb{Z}^3$. The smooth functions, $f \in \mathscr{C}^\infty(\mathbb{T}^3)$ may be written as

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{\boldsymbol{n}\in\mathbb{Z}^3} f_{\boldsymbol{n}} \cdot \mathrm{e}^{\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{x}}.$$

The Poisson bracket of two functions f, g on the 3-torus is defined as

$$\{f, g\} \equiv \varepsilon^{ijk} a^i \partial^j f \partial^k g, \tag{12}$$

once a constant vector $\boldsymbol{a} \in \mathbb{R}^3$ is given and the corresponding Poisson manifold is denoted with $\mathbb{T}^3_{\boldsymbol{a}}$. The Poisson algebra on the basis $\mathrm{e}^{\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{x}}$, $\boldsymbol{n} \in \mathbb{Z}^3$ is given by

$$\{\mathrm{e}^{\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{x}}, \mathrm{e}^{\mathrm{i}\boldsymbol{m}\cdot\boldsymbol{x}}\} = -\det(\boldsymbol{a}, \boldsymbol{m}, \boldsymbol{n})\, \mathrm{e}^{\mathrm{i}(\boldsymbol{m}+\boldsymbol{n})\cdot\boldsymbol{x}}. \tag{13}$$

The corresponding algebra, SDiff($\mathbb{T}^3$) can be expressed in terms of the Nambu bracket [4, 11, 13]:

$$\{\mathrm{e}^{\mathrm{i}\boldsymbol{n}_1\cdot\boldsymbol{x}}, \mathrm{e}^{\mathrm{i}\boldsymbol{n}_2\cdot\boldsymbol{x}}, \mathrm{e}^{\mathrm{i}\boldsymbol{n}_3\cdot\boldsymbol{x}}\} = -\mathrm{i} \times \det(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{n}_1+\boldsymbol{n}_2+\boldsymbol{n}_3)\cdot\boldsymbol{x}}. \tag{14}$$

The compactification on the torus or on the sphere can also be considered as an infrared cutoff for growing modes of incompressible flows over large distances in fluid dynamics [16].

3. Linear Nambu flows

In this work, we focus on the case of *linear* Nambu flows, which can be derived from a pair of Hamiltonians, $H_2 = \boldsymbol{a} \cdot \boldsymbol{x}$ and $H_1 = (1/2)(\boldsymbol{x}, \mathsf{B}\boldsymbol{x})$, where $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^3$ and B is a real, symmetric, 3×3 matrix. The corresponding trajectory of the linear Nambu flow (LNF) is given by

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \varepsilon^{ijk} a^j B^{kl} x^l \equiv x^l \mathsf{M}^{li}. \tag{15}$$

The solutions, given an initial condition, $x^i(0)$, lie on the intersection of the plane with normal vector, $\boldsymbol{a}$ and the quadratic surface given by $H_1 = (1/2)(\boldsymbol{x}(0), \mathsf{B}\boldsymbol{x}(0))$. We can integrate the equation of motion explicitly and find

$$\boldsymbol{x}(t) = \boldsymbol{x}(0)\, \mathrm{e}^{t\mathsf{M}}. \tag{16}$$

Since the matrix M is traceless, $\mathsf{A} \equiv \mathrm{e}^{\mathsf{M}}$ is an element of the group $SL(3, \mathbb{R})$. The converse is *not* true, i.e. every traceless matrix M (element of the Lie algebra, $sl(3, \mathbb{R})$) defines an incompressible flow, which, in general, does not admit a representation in terms of a linear and a quadratic Hamiltonian pair. If we require, in addition, that M have an eigenvector with zero eigenvalue, then it can be shown that it is of the 'Nambu form', equation (15).

It is possible to compactify the LNF on $\mathbb{T}^3$, if we consider the linear evolution equation (15), modulo the size of the torus, i.e. we take $x^i, i = 1, 2, 3$, to belong to the elementary cell, $x^i \equiv x^i + L^i$, where L^i is the length of the torus along the direction x^i. We choose our units so that $L^i = 2\pi$. The action of the matrix A on every point of $\mathbb{T}^3$ is then taken modulo 2π. These flows are known in the literature as *toral automorphisms* [17]. The motion in this case, even though the equation is linear, can be chaotic, depending on the matrix elements of A. We can check that, for LNF in $\mathbb{R}^3$, we have, essentially, a reduction to a two-dimensional phase space problem on the plane orthogonal to the vector $\boldsymbol{a}$. In the case of $\mathbb{T}^3$, if the vector has rational components, then we have a finite number of different images of the plane; if, however, the components are irrationals, then we have a truly three-dimensional evolution for the system.

Considering the algebra of all LNF, we characterize the corresponding generators by a vector, $\boldsymbol{a} \in \mathbb{R}^3$, and a symmetric, 3×3 matrix B,

$$X(\boldsymbol{a}, \mathsf{B}) = -\varepsilon^{ijk} a^j B^{kl} x^l \partial^i. \tag{17}$$

Their Lie algebra closes as follows:

$$\begin{aligned} &[X(\boldsymbol{a}_1, \mathsf{B}_1), X(\boldsymbol{a}_2, \mathsf{B}_2)] = X(\boldsymbol{a}_3, \mathsf{B}_2) + X(\boldsymbol{a}_2, \mathsf{B}_3), \\ &a_3^l = \varepsilon^{ijk} a_2^i a_1^j B_1^{kl}, \\ &B_3^{lm} = 2\varepsilon^{ijk} a_1^j B_1^{kl} B_2^{im}. \end{aligned} \tag{18}$$

Since this algebra contains a total of eight independent parameters, it can generate $SL(3,\mathbb{R})$, i.e. consecutive application of different LNF gives rise to an $SL(3,\mathbb{R})$ flow, which is not necessarily LNF.

4. The discrete phase space of linear Nambu flows

The simplest discretization of $\mathbb{T}^3_\theta$ (where $\boldsymbol{\theta} \in \mathbb{R}^3$) can be constructed by considering only points with rational coordinates, $x^i = 2\pi k^i/N$, k^1, k^2, k^3 integers modulo *N*, whose denominator is a fixed prime number, *N*. Discretization of flows is necessary in order to provide an ultraviolet cutoff to nonlinear, classical, instabilities [18]. This set forms a three-dimensional, Abelian, group, $\mathbb{T}^3_N$ under addition of coordinates modulo 2π. The linear maps, which define the evolution in this discrete phase space, are elements of $Sl(3,\mathbb{Z}_N)$, i.e. 3×3 integer matrices with entries taken modulo *N* and determinant equal to one (modulo *N*). The discrete time evolution, for any $\mathsf{A} \in SL(3,\mathbb{Z}_N)$, is given as

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n \cdot \mathsf{A}, \tag{19}$$

whose solution is

$$\boldsymbol{x}_n = \boldsymbol{x}_0 \cdot \mathsf{A}^n, \qquad n = 0, 1, 2, \ldots. \tag{20}$$

Since the group $SL(3,\mathbb{Z}_N)$ is finite, all orbits are periodic and there exist interesting special motions, which form subgroups thereof, namely, shears, rotations and dilatations (cf also below). The shears form the discrete Heisenberg–Weyl subgroup HW_N, which is the set of elements (acting on the right side of points of $\mathbb{T}^3_N$):

$$\mathsf{T}(a,b,c) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \qquad a, b, c \in \mathbb{Z}_N, \tag{21}$$

$$\begin{aligned} &\mathsf{T}(a_1,b_1,c_1)\mathsf{T}(a_2,b_2,c_2) = \mathsf{T}(a_1+a_2, b_1+b_2, c_1+c_2+b_1a_2) \\ &\qquad a_i, b_i, c_i \in \mathbb{Z}_N \qquad i = 1, 2, \end{aligned} \tag{22}$$

an inverse element

$$\mathsf{T}^{-1}(a,b,c) = \mathsf{T}(-a,-b,-c+ab) \tag{23}$$

and center (equal to Z_N) generated by the element $\Omega = T(0,0,1)$. The commutation relations of two elements are given by

$$\mathsf{T}(a_1,b_1,c_1)\mathsf{T}(a_2,b_2,c_2) = \Omega^{b_1a_2 - b_2a_1}\mathsf{T}(a_2b_2c_2)\mathsf{T}(a_1,b_1,c_1). \tag{24}$$

If we denote the generators of the one-parameter subgroups by P and Q,

$$P = \mathsf{T}(1,0,0), \qquad Q = \mathsf{T}(0,1,0), \tag{25}$$

we obtain the Heisenberg–Weyl commutation relation [19]

$$QP = \Omega PQ, \qquad \Omega P = P\Omega, \qquad \Omega Q = Q\Omega, \tag{26}$$

which together with the periodicity properties,

$$Q^N = P^N = \Omega^N = I, \tag{27}$$

define the discrete Heisenberg–Weyl group, HW_N. The general element equation (21) can be written as

$$\mathsf{T}(a,b,c) = \Omega^c P^a Q^b. \tag{28}$$

The subgroups mentioned previously have the following matrix realizations, the dilatations:

$$\mathsf{D}(a,b) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix}. \tag{29}$$

The rotations, which form the discrete subgroup, $SO(3,\mathbb{Z}_N)$, preserves the norm, $(x^1)^2 + (x^2)^2 + (x^3)^2 \bmod N$ and are generated by the following matrices:

$$\mathsf{R}_1(a_1,b_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & b_1 \\ 0 & -b_1 & a_1 \end{pmatrix}, \qquad \mathsf{R}_2(a_2,b_2) = \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ -b_2 & 0 & a_2 \end{pmatrix},$$
$$\mathsf{R}_3(a_3,b_3) = \begin{pmatrix} a_3 & b_3 & 0 \\ -b_3 & a_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{30}$$

with $a_i^2 + b_i^2 \equiv 1 \bmod N$, for $i = 1,2,3$, through the group law $\mathsf{R} = \mathsf{R}_1(a_1,b_1)\mathsf{R}_2(a_2,b_2)\mathsf{R}_3(a_3,b_3)$.

Another interesting subgroup of $SL(3,\mathbb{Z}_N)$ is the discrete Lorentz group, $SO(2,1,\mathbb{Z}_N)$, where, in R_2 and R_3 we replace $-b_2$ and $-b_3$ with b_2 and b_3, respectively. These elements then preserve the norm $(x^1)^2 - (x^2)^2 - (x^3)^2 \bmod N$.

Since N is prime, $\mathbb{Z}_N$ is a finite algebraic field, and there exists a primitive element, g, whose successive powers generate all the elements of the field. If $N = 4k \pm 1$, then the subgroups generated by each R_i are cyclic, of order $4k$ and they contain the three duality matrices (Fourier transforms) for each of the phase space planes (12, 23, 31)[20].

To characterize discrete LNFs we must determine the form of the elements of $SL(3,\mathbb{Z}_N)$ which leave invariant a given vector, $\boldsymbol{a} \in \mathbb{T}_N^3$, a (left) eigenvector of the evolution matrix, A, with eigenvalue unity

$$\boldsymbol{a} = \boldsymbol{a} \cdot \mathsf{A}. \tag{31}$$

Rotations and translations have, indeed, this property, since they do leave certain vectors invariant, whereas certain dilatations do not.

For any such vector, $\boldsymbol{a} = (a_1,a_2,a_3)$, $a_i \in \mathbb{Z}_N$, condition (31) allows us to solve for the elements A_{31}, A_{32}, A_{33}, if A_{33} is different from zero. The little group of $\boldsymbol{a}$ is an $SL(2,\mathbb{Z}_N)$ subgroup. Indeed, the evolution equation, (19), becomes

$$[\boldsymbol{a}\times\boldsymbol{x}]_{n+1} = [\boldsymbol{a}\times\boldsymbol{x}]_n \begin{pmatrix} A_{22} - \frac{a_2}{a_3}A_{23} & -\left(A_{21} - \frac{a_1}{a_3}A_{23}\right) & \frac{A_{21}a_2 - A_{22}a_1}{a_3} \\ -\left(A_{12} - \frac{a_2}{a_3}A_{13}\right) & A_{11} - \frac{a_1}{a_3}A_{13} & -\frac{A_{11}a_2 - A_{12}a_1}{a_3} \\ 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

This equation implies that the component of any vector $\boldsymbol{x}$, parallel to $\boldsymbol{a}$, is preserved under this evolution, while the components that lie on the plane perpendicular to $\boldsymbol{a}$ and are represented by $\boldsymbol{a} \times \boldsymbol{x}$, evolve under the 2×2 matrix:

$$\widetilde{\mathsf{A}} \equiv \begin{pmatrix} A_{11} - \frac{a_1}{a_3} A_{13} & A_{21} - \frac{a_1}{a_3} A_{23} \\ A_{12} - \frac{a_2}{a_3} A_{13} & A_{22} - \frac{a_2}{a_3} A_{23} \end{pmatrix} \tag{33}$$

(it is noteworthy that $\widetilde{\mathsf{A}}$ is the inverse of the 2×2 block in equation (32)!). It is quite straightforward to check that the determinant of $\widetilde{\mathsf{A}}$ is equal to 1 if $\boldsymbol{a} = \boldsymbol{a} \cdot \mathsf{A}$; thus, for, any such $\mathsf{A} \in SL(3, \mathbb{Z}_N)$, we have a mapping to an $\widetilde{\mathsf{A}} \in SL(2, \mathbb{Z}_N)$, which is the little subgroup of A which leaves invariant the vector $\boldsymbol{a}$. This mapping is a group homomorphism, $\widetilde{\mathsf{A}}\widetilde{\mathsf{B}} = \widetilde{\mathsf{AB}}$.

This mapping will be useful for the quantization of LNFs in $SL(3, \mathbb{Z}_N)$.

5. The non-commutative 3-torus

Non-commutative tori play an important role in non-commutative geometry [21, 24], in M-theory matrix models[25] and quantum Hall effect [26, 27]. In the present context, we need the description of the non-commutative 3-torus, which is appropriate for the study of quantization of linear Nambu flows.

Let us begin by recalling that it is possible to embed the Heisenberg–Weyl algebra for one degree of freedom:

$$[x_1, x_2] = \mathrm{i}\hbar I, \tag{34}$$

in the three-dimensional non-commutative 3-space $\mathbb{R}^3_\theta$,

$$[x^i, x^j] = \mathrm{i}\hbar\epsilon^{ijk}\theta^k, \qquad i, j, k = 1, 2, 3, \qquad \boldsymbol{\theta} \in \mathbb{R}^3, \tag{35}$$

so that the two-dimensional quantum phase space is defined by the Casimir [13]:

$$C = \boldsymbol{\theta} \cdot \boldsymbol{x}. \tag{36}$$

We can compactify this algebra by considering the algebra of the group elements :

$$T_{\boldsymbol{a}} = \mathrm{e}^{\mathrm{i}\boldsymbol{a}\cdot\boldsymbol{x}}, \qquad \boldsymbol{a} \in \mathbb{R}^3. \tag{37}$$

They satisfy

$$T_{\boldsymbol{a}} T_{\boldsymbol{b}} = \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}\det(\boldsymbol{a},\boldsymbol{b},\boldsymbol{\theta})} T_{\boldsymbol{a}+\boldsymbol{b}}. \tag{38}$$

These imply

$$[T_{\boldsymbol{a}}, T_{\boldsymbol{b}}] = -2\mathrm{i}\sin\left(\frac{\hbar}{2}\det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\theta})\right) T_{\boldsymbol{a}+\boldsymbol{b}}. \tag{39}$$

If the quantization and deformation parameters satisfy

$$\hbar\boldsymbol{\theta} = \frac{2\pi}{N}(k_1, k_2, k_3), \qquad k_i \in \mathbb{Z}_N \quad i = 1, 2, 3, \tag{40}$$

then the Hilbert space of the objects becomes an infinite set of identical copies of the Hilbert space, $\mathscr{H}_N$, of dimension N. In this space, let us define the discrete non-commutative 3-torus algebra as the set generated by three $N \times N$ unitary matrices Q_i, $i = 1, 2, 3$, satisfying (for fixed k_i mod N, $i = 1, 2, 3$)

$$Q_3 Q_2 = \omega^{k_1} Q_2 Q_3, \qquad Q_1 Q_3 = \omega^{k_2} Q_3 Q_1 \qquad Q_2 Q_1 = \omega^{k_3} Q_1 Q_2, \tag{41}$$

with $k_i \in \mathbb{Z}_N$ and $\omega \equiv \exp(2\pi\mathrm{i}/N)$.

The magnetic translation operators can be defined as

$$J_{\boldsymbol{m}} = \omega^{\frac{1}{2}(k_3 m_1 m_2 + k_1 m_2 m_3 - k_2 m_3 m_1)} Q_1^{m_1} Q_2^{m_2} Q_3^{m_3}. \tag{42}$$

The phase is crucial and has to be chosen so that we have

$$J_{\boldsymbol{m}}^{\dagger} = J_{-\boldsymbol{m}}, \qquad \boldsymbol{m} \in \mathbb{Z}_N^3. \tag{43}$$

We then also find that

$$J_{\boldsymbol{m}} J_{\boldsymbol{n}} = \omega^{-\frac{1}{2}\det(\boldsymbol{k},\boldsymbol{m},\boldsymbol{n})} J_{\boldsymbol{m}+\boldsymbol{n}}, \qquad \boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}_N^3. \tag{44}$$

We can see that these commutation relations imply the existence of central element of the algebra, $C_T = Q_1^{k_1} Q_2^{k_2} Q_3^{k_3}$. If the representation is irreducible, then C must be proportional to the unit element, up to a phase, $C_T = \omega^{c_N} \times I$, with $c_N \in \mathbb{Z}_N$. Consider now the elements $J_{\alpha\boldsymbol{k}}$, with $\alpha \in \mathbb{Z}_N$. It is easy to check that they are pure phases:

$$J_{\alpha\boldsymbol{k}} = \omega^{\frac{\alpha^2}{2} k_1 k_2 k_3 + \alpha(c_N - k_1 k_2 k_3)} \times I. \tag{45}$$

Therefore, the N^3 magnetic translations are divided into a subgroup of N phases and a set of N^2 unitary matrices $J_{\boldsymbol{m}}$, where $\boldsymbol{m}$ is orthogonal to $\boldsymbol{k}$. This structure resembles that of the discrete Heisenberg–Weyl group, HW_N. The magnetic translation operators thus depend on the vector $\boldsymbol{k}$, and we shall henceforth explicitly highlight this by writing them as $J_{\boldsymbol{m}}(\boldsymbol{k})$. The commutation relations between $J_{\boldsymbol{m}}(\boldsymbol{k})$ and $J_{\boldsymbol{m}}(\boldsymbol{k}')$ can be computed once we shall establish the relation between these magnetic translations and the Heisenberg–Weyl generators.

The classical action of the discrete map $\mathsf{A} \in SL(3, \mathbb{Z}_N)$ on the points $\boldsymbol{m} = (m_1, m_2, m_3)$ of the torus T_N^3 was reduced in the previous section to the action of $\widetilde{\mathsf{A}} \in SL(2, \mathbb{Z}_N)$ on the points $\widetilde{\boldsymbol{m}} \equiv (a_3 m_1 - a_1 m_3, a_3 m_2 - a_2 m_3)$ of the plane orthogonal to $\boldsymbol{a}$.

If we restrict $\boldsymbol{m}$ to this plane, we obtain

$$\widetilde{\boldsymbol{m}} = (m_1, m_2) T(\boldsymbol{a}) \equiv (m_1, m_2) \begin{pmatrix} \dfrac{1 - a_2^2}{a_3} & \dfrac{a_1 a_2}{a_3} \\ \dfrac{a_1 a_2}{a_3} & \dfrac{1 - a_1^2}{a_3} \end{pmatrix}, \tag{46}$$

where we have assumed that $a_1^2 + a_2^2 + a_3^2 \equiv 1 \bmod N$. In this case $T(\boldsymbol{a})$ is an element of $SL(2, \mathbb{Z}_N)$.

Upon quantization on this discrete two-dimensional phase space, we should employ the discrete Heisenberg–Weyl group, generated by the clock and shift $N \times N$ matrices Q, P [31]:

$$Q_{k,l} = \omega^k \delta_{k,l}, \qquad P_{k,l} = \delta_{k,l+1}, \qquad k, l \in \mathbb{Z}_N, \tag{47}$$

which satisfy

$$QP = \omega P Q. \tag{48}$$

The corresponding two-dimensional magnetic translations defined by

$$J_{r,s} = \omega^{\frac{rs}{2}} P^r Q^s, \qquad r, s \in \mathbb{Z}_N, \tag{49}$$

satisfy the relations

$$\begin{aligned} J_{r,s} J_{r\prime,s\prime} &= \omega^{\frac{r\prime s - r s\prime}{2}} J_{r+r\prime,s+s\prime}, \\ [J_{r,s}]^{\dagger} &= J_{-r,-s}, \end{aligned} \tag{50}$$

where $r, s, r', s' \in \mathbb{Z}_N$.

We now identify the points of the torus $\mathbb{T}_N^2$ on which the map $\widetilde{\mathsf{A}}$ acts with the indices, (r, s) of the two-dimensional magnetic translations, $J_{r,s}$ through

$$r \equiv a_3 m_1 - a_1 m_3, \qquad s \equiv a_3 m_2 - a_2 m_3, \tag{51}$$

as

$$J_{\boldsymbol{m}}(\boldsymbol{k}) = J_{(m_1,m_2)T(\boldsymbol{a})}. \tag{52}$$

This *ansatz* implies the relations

$$\begin{aligned} J_{1,0,0} &= Q_1 = J_{a_3,0} = P^{a_3}, \\ J_{0,1,0} &= Q_2 = J_{0,a_3} = Q^{a_3}, \\ J_{0,0,1} &= Q_3 = J_{-a_1,-a_2} = \omega^{\frac{a_1 a_2}{2}} P^{-a_1} Q^{-a_2}, \end{aligned} \tag{53}$$

and from the commutation relations of the operators Q_1, Q_2, Q_3, we find

$$k_1 = a_1 a_3, \qquad k_2 = a_2 a_3, \qquad k_3 = a_3^2. \tag{54}$$

This identification also fixes the phase, c_N, of the Casimir in equation (45) as

$$c_N = \frac{k_1 k_2 k_3}{2} = \frac{a_1 a_2 a_3^4}{2}. \tag{55}$$

Thus the N phases have been eliminated and only magnetic translations in the plane orthogonal to the vector $\boldsymbol{a}$ survive. It is possible to represent the algebra of equation (41) by 3×3 matrices of $SL(3,\mathbb{Z}_N)$ substituting in equation (53) the $N \times N$ matrices P, Q, $\omega \cdot I$ by the matrices $P = \mathsf{T}(1,0,0)$, $Q = \mathsf{T}(0,1,0)$, $\Omega = \mathsf{T}(0,0,1)$ of the 3×3 Heisenberg–Weyl group in equation (28).

In order to generate the full magnetic translation group of the three-dimensional, discrete torus, we must consider three, mutually orthogonal, planes and their corresponding $J_{\boldsymbol{m}}(\boldsymbol{a})$'s. For example, $\boldsymbol{a} = (1,0,0)$, $\boldsymbol{a} = (0,1,0)$ and $\boldsymbol{a} = (0,0,1)$. Starting from the 1–2 plane and applying discrete rotations of $SO(3,\mathbb{Z}_N)$, cf equation (30), we can generate the other two. To construct the corresponding $J_{\boldsymbol{m}}$'s for the 2–3 and 3–1 planes, we must construct the corresponding unitary, $N \times N$ operators, $U(\mathsf{R}_{1,2,3})$. This remains to be done.

In the following section, we shall apply the above results for the quantization of the classical Nambu mechanics in the case of linear flows, for a fixed plane.

6. Quantization of linear Nambu flows on a discretized 3-torus

There is a long-standing problem on how to quantize Nambu mechanics, and there are various proposals, which, however, do not respect the fundamental properties of the classical Nambu bracket, such as Leibniz and the fundamental identity [2, 3, 5–7, 11].

Quantization of the classical dynamics, $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n \mathsf{A}$, for $\mathsf{A} \in SL(3,\mathbb{Z}_N)$, means constructing a unitary operator, $U(\mathsf{A})$ as a $N \times N$ unitary matrix, that satisfies

$$U^{\dagger}(\mathsf{A}) J_{\boldsymbol{m}} U(\mathsf{A}) = J_{\boldsymbol{m}\cdot\mathsf{A}} \tag{56}$$

in the basis of the complete set of three-dimensional magnetic translations of the non-commutative 3-torus. This would realize the N-dimensional *metaplectic* representation of the double cover of $SL(3,\mathbb{Z}_N)$. For rigorous mathematical results pertaining to the metaplectic representation of the double cover of $SL(3,\mathbb{F})$, for $\mathbb{F}$ a local field, we refer to the literature [28]; for $\mathbb{F} = \mathbb{R}$, cf [29].

The results of the previous sections allow us to do this, for the case of LNFs. Indeed, we found that, for those flows, the classical evolution equation, $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n \cdot \mathsf{A}$ may be written in the form of

$$[\boldsymbol{a} \times \boldsymbol{x}_{n+1}] = [\boldsymbol{a} \times \boldsymbol{x}_n] \begin{pmatrix} \widetilde{\mathsf{A}}^{-1} & \widetilde{\boldsymbol{a}} \\ \boldsymbol{0} & 0 \end{pmatrix},$$

where $\widetilde{\mathsf{A}}$ is given by equation (33) and the vector $\widetilde{\boldsymbol{a}}^T \equiv ((A_{21}a_2 - A_{22}a_1)/a_3, -(A_{11}a_2 - A_{12}a_1)/a_3)$ and thus the interesting dynamical variables are the combinations $[\boldsymbol{a}\times\boldsymbol{x}_n]_1 \equiv a_2x_3 - a_3x_2$ and $[\boldsymbol{a}\times\boldsymbol{x}_n]_2 \equiv a_3x_1 - a_1x_3$; the other component may be expressed as a linear combination of these. Since $\widetilde{\mathsf{A}} \in SL(2,\mathbb{Z}_N)$ when $\mathsf{A} \in SL(3,\mathbb{Z}_N)$ and $\boldsymbol{a} = \boldsymbol{a}\cdot\mathsf{A}$, we know how to construct the unitary operator, $U(\widetilde{\mathsf{A}})$, that realizes the metaplectic representation for $\widetilde{\mathsf{A}}$. Furthermore, we may verify that $\widetilde{\mathsf{A}}\cdot\widetilde{\mathsf{B}} = \widetilde{\mathsf{AB}}$, for any two matrices $\mathsf{A},\mathsf{B} \in SL(3,\mathbb{Z}_N)$ that have $\boldsymbol{a}$ as a common eigenvector, with eigenvalue unity, $\boldsymbol{a} = \boldsymbol{a}\cdot\mathsf{A}$, $\boldsymbol{a} = \boldsymbol{a}\cdot\mathsf{B}$. So we can write the following commuting, diagram:

$$\begin{array}{ccc} \mathsf{A} & \longrightarrow & \widetilde{\mathsf{A}} \\ \downarrow & & \downarrow \\ U(\mathsf{A}) & \longrightarrow & U(\widetilde{\mathsf{A}}). \end{array} \tag{57}$$

To construct the corresponding (unitary) evolution operator, $U(\mathsf{A})$, we thus use the metaplectic representation of $SL(2,\mathbb{Z}_N)$ for the 'reduced' 2×2 matrix, $\widetilde{\mathsf{A}}$ of equation (33) which satisfies

$$U(\widetilde{\mathsf{A}})^\dagger J_{r,s} U(\widetilde{\mathsf{A}}) = J_{(r,s)\widetilde{\mathsf{A}}}, \tag{58}$$

and is given, for any element

$$\widetilde{\mathsf{A}} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}_N),$$

by the expression [30, 31]

$$U(\widetilde{\mathsf{A}})_{k,l} = \frac{\sigma_N(c)}{\sqrt{N}}\omega^{\frac{ak^2-2kl+dl^2}{2c}}, \tag{59}$$

where

$$\sigma_N(c) \equiv \frac{1}{\sqrt{N}}\sum_{r=0}^{N-1}\omega^{c\cdot r^2}$$

is the Gauss sum [20].

The prefactor, $\sigma_N(c)$, ensures that this representation is not only projective, but *faithful*, i.e., for *any* two matrices, $\widetilde{\mathsf{A}}$ and $\widetilde{\mathsf{B}}$, elements of $SL(2,\mathbb{Z}_N)$, we have that $U(\widetilde{\mathsf{AB}}) = U(\widetilde{\mathsf{A}})U(\widetilde{\mathsf{B}})$.

Having thus shown that, for LNFs, the interesting dynamics takes place on a plane perpendicular to the vector $\boldsymbol{a}$ that enters in the definition of $H_2 \equiv \boldsymbol{a}\cdot\boldsymbol{x}$ we can understand why the construction of the unitary operator $U(\mathsf{A}) \equiv U(\widetilde{\mathsf{A}})$ amounts to a quantization of discrete position and momenta: from the classical vectors $\boldsymbol{m}$ and $\boldsymbol{a}$ we construct the corresponding position and momentum variables $r \equiv a_3m_1 - a_1m_3$ and $s \equiv a_3m_2 - a_2m_3$, respectively. These evolve using the operator $\widetilde{\mathsf{A}}$, while the N-dimensional,complex vector (wavefunction), in the position representation, depends on $r = 0, 1, 2, \ldots, N-1$ and evolves according to $U(\widetilde{\mathsf{A}})$. From this operator, we may calculate the average value(s) of physical observables, as well as correlation functions of the flow, using standard quantum-mechanical techniques. For physically interesting subgroups of $SL(3,\mathbb{Z}_N)$, mentioned in section 4, we may find the eigenstates and eigenvalues of $U(\widetilde{\mathsf{A}})$ explicitly. This will be reported elsewhere.

7. Conclusions

We have constructed the quantization of Nambu mechanics, for the case of linear flows on the discrete three-dimensional torus considered as a phase space. Our method proposes also a scheme for the quantization of the Nambu 3-bracket as the algebra of the foliation of the non-commutative 3-torus by a family of Heisenberg–Weyl groups of all of its linear two-dimensional subspaces (non-commutative 2-tori). The key idea was to use the metaplectic representation of $SL(3, \mathbb{Z}_N)$, induced by that of $SL(2, \mathbb{Z}_N)$. Considering potential applications, our method can be extended to the full set of discrete linear flows, not necessarily of the Nambu type (not having invariant two-dimensional subspaces). This will lead to the quantization of strong arithmetic chaos [22] on the discrete 3-torus and can be used as a toy model for the quantization of turbulent maps [23]. Considering the coset, $SL(3, \mathbb{Z}_N)/SO(3, \mathbb{Z}_N)$, we can construct the corresponding quantum coherent states (discrete orthogonal wavelets) for the wavelet transform of three-dimensional objects (quantum tomography in the spirit of [30, 32]. Another possible direction is the study of discrete non-commutative solitons in three dimensions using non-dispersive t'Hooft states [33]. Concluding we believe that the proposed framework of quantization for Nambu mechanics will lead to new insights for the quantization of the volume-preserving diffeomorphism group in three dimensions.

References

[1] Bagger J and Lambert N 2007 *Phys. Rev.* D **75** 045020 (arXiv:hep-th/0611108)
[2] Nambu Y 1973 *Phys. Rev.* D **7** 2405
[3] Takhtajan L 1994 *Commun. Math. Phys.* **160** 295 (arXiv:hep-th/9301111)
[4] Hoppe J 1997 *Helv. Phys. Acta* **70** 302 (arXiv:hep-th/9602020)
[5] Awata H, Li M, Minic D and Yoneya T 2001 *J. High Energy Phys.* JHEP02(2001)013 (arXiv:hep-th/9906248)
[6] Minic D and Tze H C 2002 *Phys. Lett.* B **536** 305 (arXiv:hep-th/0202173)
[7] Kawamura Y 2003 *Prog. Theor. Phys.* **109** 153 (arXiv:hep-th/0207054)
[8] Zachos C K and Curtright T 2004 *Czech. J. Phys.* **54** 1393 (arXiv:math-ph/0408012)
[9] Gustavsson A 2007 Algebraic structures on parallel M2-branes arXiv:0709.1260 [hep-th]
[10] Papadopoulos G 2008 *J. High Energy Phys.* JHEP05(2008)054 (arXiv:0804.2662 [hep-th])
Bonelli G, Tanzini A and Zabzine M 2009 *Phys. Lett.* B **672** 390 (arXiv:0807.5113 [hep-th])
[11] Ho P M, Hou R C and Matsuo Y 2008 *J. High Energy Phys.* JHEP06(2008)020 (arXiv:0804.2110 [hep-th])
[12] Ho P M and Matsuo Y 2008 *J. High Energy Phys.* JHEP06(2008)105 (arXiv:0804.3629 [hep-th])
Ho P M, Imamura Y and Matsuo Y 2008 *J. High Energy Phys.* JHEP07(2008)003 (arXiv:0805.1202 [hep-th])
[13] Axenides M and Floratos E G 2008 Nambu-Lie 3-algebras on fuzzy 3-manifolds arXiv:0809.3493 [hep-th]
[14] Marsden J and Weinstein A 1983 *Physica* **7D** 305
[15] Lamb H 1932 *Hydrodynamics* (Cambridge: Cambridge University Press) p 248
[16] Bajer K, Bajer H K and Moffatt H K 1990 *J. Fluid Mech.* **212** 337
[17] Arnold V I and Khesin B A 1998 *Topological Methods in Hydrodynamics* (Berlin: Springer)
[18] Rajeev S G 2005 *Int. J. Mod. Phys.* A **20** 6122
[19] Schwinger J 1970 *Quantum Kinematics and Dynamics* (New York: Benjamin)
[20] Lang S L 1970 *Analytic Number Theory* (Reading, MA: Addison-Wesley)
[21] Rieffel M 1988 *Can. J. Math.* **40** 257
Manin Yu 2000 Theta functions, quantum tori and Heisenberg groups arXiv:math.AG/0011197
Schwarz A 2001 Theta functions on non-commutative tori arXiv:math/0107186
[22] Percival I C and Vivaldi F 1987 *Physica* D **25** 105
[23] Ugalde E and Lima R 1996 *Physica* D **95** 144
[24] Balachandran A P, Kurkcuoglu S and Vaidya S 2007 *Lectures on Fuzzy and Fuzzy SUSY Physics* (Singapore: World Scientific)
[25] Connes A, Douglas M R and Schwarz A S 1998 *J. High Energy Phys.* JHEP02(1998)003 (arXiv: hep-th/9711162)
[26] Khveshchenko D and Wiegmann P 1989 *Phys. Lett.* B **225** 279
[27] Bedos E 1999 *J. Geom. Phys.* **30** 204
[28] Flicker Y, Kazhdan D and Savin G 1990 Explicit realization of a metaplectic representation *J. Anal. Math.* **55** 17
[29] Torasso P 1983 *Acta Math.* **150** 153
[30] Athanasiu G G and Floratos E G 1995 *Phys. Lett.* B **352** 105
Athanasiu G G and Floratos E G 1994 *Nucl. Phys.* B **425** 343
Floratos E G and Nicolis S 1998 *J. Phys. A: Math. Gen.* **31** 3961 (arXiv:hep-th/9508111)

[31] Athanasiu G G, Floratos E G and Nicolis S 1996 *J. Phys. A: Math. Gen.* **29** 6737 (arXiv:hep-th/9509098)
Athanasiu G G, Floratos E G and Nicolis S 1998 *J. Phys. A: Math. Gen.* **31** L655 (arXiv:math-ph/9805012)

[32] Coen S 1999 *Phys. Rev. Lett.* **83** 2494

[33] 't Hooft G 1990 *Nucl. Phys.* B **342** 471
Floratos E G and Leontaris G 1997 *Phys. Lett.* B **412** 35 (arXiv:hep-th/9706156)
Floratos E G and Nicolis S 2002 Non-commutative solitons in finite quantum mechanics *Nucl. Phys. Proc. Suppl.* **119** 947 (arXiv:hep-lat/0209032)